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Phase Separations in Ising Model with Free Boundary Condition

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We consider the problem of phase separations in Ising model with free boundary condition. We prove that a typical configuration has just one open contour λ which separates V into two parts which are occupied by the opposite phases. λ is the shortest possible contour compatible with the condition that V is divided by λ into two regions of area $\rho|V|$ and $(1-\rho)|V|$.

KEY WORDS: Ising model; Gibbs measure; phase separation; phase transition; contour; conditional Gibbs measure.

1. INTRODUCTION

In this paper we consider the problem of phase separations in two-dimensional Ising model with free boundary condition. The result with this boundary condition is different from the one with pure boundary condition obtained by Minlos and Sinai.[1,2]

They showed the following theorem: under the condition that the number of minus spins in the square V is fixed to be $\rho|V|$ and $(+)$-boundary condition, the following statements are satisfied asymptotically with probability 1 as $|V| \to \infty$:

$$\text{(i)} \qquad \big||\Theta_-^{\max}| - \rho|V|\big| < C_1(\beta)|V|^{3/4} \qquad [C_1(\beta)\downarrow 0 \text{ as } \beta \to \infty]$$

$$\text{(ii)} \qquad \big||\partial\Theta_-^{\max}| - 4\rho^{1/2}|V|^{1/2}\big| < C_2(\beta)|V|^{1/2} \qquad [C_2(\beta)\downarrow 0 \text{ as } \beta \to \infty]$$

where $\Theta_-^{\max}$ is the connected component of $(-)$-spins with maximal area. This theorem means that the typical configuration has just one "nearly

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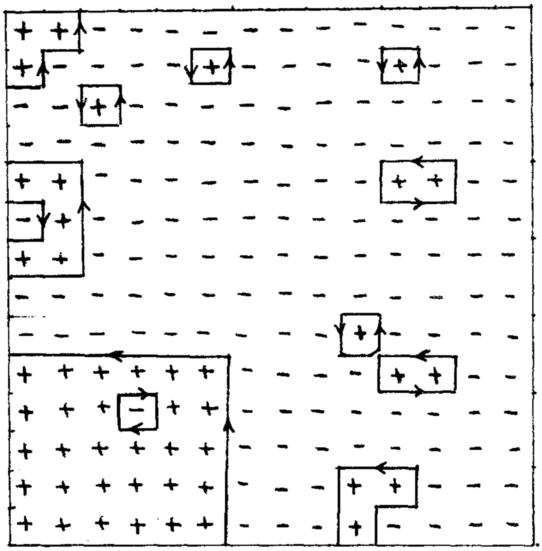

Fig. 1.

square" block of minus spins with the size of $\rho^{1/2}|V|^{1/2}$ in the "sea" of plus spins asymptotically with probability 1 as $V \uparrow Z^2$.

On the other hand, this fact does not hold under free boundary condition and the following conjecture was given by Gallavotti[3]: a typical configuration in V has just one phase separating line λ which divides V into two regions V_+ and V_- occupied by (+)-phase and (−)-phase, respectively, and should be the shortest one under the condition that the area of V_- is nearly $\rho|V|$. (See Fig. 1.)

We prove this conjecture positively with respect to the conditional Gibbs measure. In the paper of Minlos and Sinai, the number of minus spins in V is fixed, but we use the more weak condition on the number of minus spins. We describe the conjecture of Gallavotti more precisely and give the definition of conditional Gibbs measure in Section 2. We give the rigorous statement of our result in Section 3 and give the proof in Section 6. In Section 4 we prove the key lemma for the proof.

2. ISING MODEL WITH FREE BOUNDARY CONDITION

First we give the definition of Gibbs measure of two-dimensional Ising model with free boundary condition.

Let V be the square in Z^2. Put $\Omega_V = \{+1, -1\}^V$ and $\mathfrak{B}_V = \sigma\{\omega(t); t \in V\}$. For a given configuration, $\xi \in \Omega_V$, we draw a unit line segment perpendicular to each bond at the center if it joins different kinds of spins. Then these segments form lines (rectilinear curves). We attach to each line Γ the orientation along which we see plus spins on the left side, and put $\bar{\Gamma} = (\Gamma, +)$ or $(\Gamma, -)$ according as the orientation of Γ is anticlockwise or clockwise. Thus we have the family of open contours and closed contours $\{\bar{\Gamma} = (\Gamma, \pm)\}$ for each ξ. It is clear that there is a one-to-one correspondence between the configuration and the family of contours $(\bar{\Gamma}_1, \ldots, \bar{\Gamma}_s, \bar{\Delta}_1, \ldots, \bar{\Delta}_k)$, where $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_s$ are closed contours and $\bar{\Delta}_1, \ldots, \bar{\Delta}_k$ are open contours.

The Gibbs measure of Ising model with free boundary condition is defined by the following probability measure on $(\Omega_V, \mathfrak{B}_V)$:

$$P_V(\xi) = Z_V^{-1}\exp\left[-\beta\left(\sum_{i=1}^{s}|\Gamma_i| + \sum_{j=1}^{k}|\Delta_j|\right)\right]$$

$$\xi = \left(\bar{\Gamma}_1, \ldots, \bar{\Gamma}_s, \bar{\Delta}_1, \ldots, \bar{\Delta}_k\right)$$

where β^{-1} is proportional to the temperature.

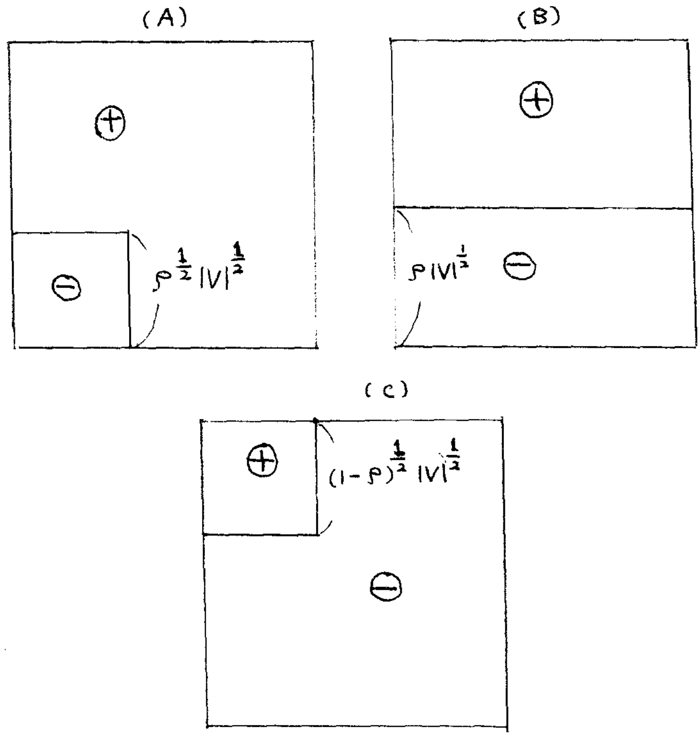

Fig. 2.

Next we describe the conjucture of Gallavotti more precisely. When the number of minus spins in square V is fixed to be $\rho|V|$, we have three types of phase separations according to the value of ρ. When $0 < \rho < 1/4$, the length $|\lambda|$ of phase separating line λ is "nearly" $2\rho^{1/2}|V|^{1/2}$ with some fluctuations, and $(-)$-phase Θ_- is "nearly square" with the size of $\rho^{1/2}|V|^{1/2}$ with some fluctuations. (See Fig. 2A.) When $1/4 < \rho < 3/4$, $|\lambda|$ is "nearly" $|V|^{1/2}$, λ starts from one side of square V and ends in the opposite side, and Θ_- is "nearly rectangle" whose sides are $\rho|V|^{1/2}$ and $|V|^{1/2}$. (See Fig. 2B.) When $3/4 < \rho < 1$, $|\lambda|$ is "nearly" $2(1-\rho)^{1/2}|V|^{1/2}$ and $(+)$-phase Θ_+ is "nearly square" with the size of $(1-\rho)^{1/2}|V|^{1/2}$. (See Fig. 2C.) We prove this conjecture positively with respect to the following conditional Gibbs measure $P_{V,\rho}(\cdot)$.

Let $N^-(\xi; V)$ be the number of minus spins in V under the configuration $\xi \in \Omega_V$ and let $g(\beta)$ be the function of β satisfying $g(\beta)\downarrow 0$ and $\exp(4\beta)g(\beta)\downarrow 0$ as $\beta \to \infty$.

Put

$$N_\rho^- = \{\xi \in \Omega_V; |N^-(\xi; V) - \rho|V|| < g(\beta)|V|\}$$

where $0 < \rho < 1$. Then the conditional Gibbs measure is given by

$$P_{V,\rho}(\cdot) = P_V(\cdot \mid N_\rho^-)$$

3. STATEMENT OF RESULTS

Before describing our results we prepare some terminologies. Since each configuration ξ can be identified with a family of contours, we write $\bar{\Gamma} \in \xi$ if $\bar{\Gamma}$ is contained in ξ as a contour. We also call open contour $\bar{\Gamma}$ "transversal" if Γ starts from one side of V and ends in the opposite side. So the length of transversal contour is greater than $|V|^{1/2}$.

For each configuration $\xi \in \Omega_V$, we denote the totality of transversal open contours by ξ^2_{open} and denote others by ξ^1_{open}. Let $c_0 > 0$, $\bar{\Gamma}$ is called c_0-large if $|\Gamma| > c_0 \ln|V|$, and others are called c_0-small. We call c_0-large open contours and c_0-large closed contours which are not surrounded by any c_0-small closed contours "phase boundary" and denote the totality of them by Λ. Next we give the definition of $(+)$-phase and $(-)$-phase. Let $\xi^1_{c_0,\text{open}}$ be the totality of c_0-small contours $\bar{\Gamma}$ in ξ^1_{open}. Let $R_{c_0}(\xi)$ be the set of those sites which are surrounded by some contour in $\xi^1_{c_0,\text{open}}$. We denote the area of $R_{c_0}(\xi)$ by $|R_{c_0}(\xi)|$. From the definition of c_0-smallness we have

$$|R_{c_0}(\xi)| < 2c_0|V|^{1/2}\ln|V|$$

Thus V is divided into three parts $(+)$-phase Θ_+, $(-)$-phase Θ_-, and

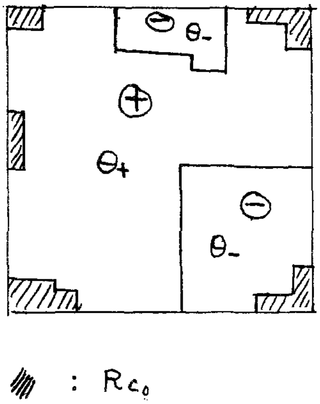

Fig. 3.

$R_{c_0}(\xi)$. (See Fig. 3.) We also denote the connected components in $\Theta_\pm$ with maximal area by $\Theta_\pm^{\max}$.

Now let us state our results.

Theorem. Let us fix $\rho \in (0,1/4) \cup (1/4,3/4) \cup (3/4,1)$ and take the value of β sufficiently large. Then the following properties are valid as to the area of $\Theta_-^{\max}$ and the number of $(\pm)$ spins in $\Theta_\pm$:

(I) $$\lim_{V \uparrow Z^2} P_{V,\rho}\left(\left||\Theta_-^{\max}| - \rho|V|\right| > k(\beta)|V|\right) = 0$$

(II) $$\lim_{V \uparrow Z^2} P_{V,\rho}\left(\left|N_\pm(\xi;\Theta_\pm) - \rho^{**}(\beta)|\Theta_\pm|\right| > h(\beta)|V|^{3/4}\right) = 0$$

Here $k(\beta) = (5k_0/3\beta)^2$, $k_0 = 4/c_0 + 1$, $\rho^{**}(\beta) = 1 - \rho^*(\beta)$, and $h(\beta)$, $\rho^*(\beta) \downarrow 0$ as $\beta \to \infty$.

Furthermore the following properties are satisfied as to the length of phase boundary $|\Lambda|$ according to the value of ρ:

(III-i) If $0 < \rho < 1/4$, then we have

$$\lim_{V \uparrow Z^2} P_{V,\rho}\left(|\Lambda| > (2\rho^{1/2} + k_0/\beta)|V|^{1/2}\right) = 0$$

(III-ii) If $1/4 < \rho < 3/4$, then we have

$$\lim_{V \uparrow Z^2} P_{V,\rho}\left(|\Lambda| > (1 + k_0/\beta)|V|^{1/2}\right) = 0$$

(III-iii) If $3/4 < \rho < 1$, then we have

$$\lim_{V \uparrow Z^2} P_{V,\rho}\left(|\Lambda| > \left[2(1-\rho)^{1/2} + k_0/\beta\right]|V|^{1/2}\right) = 0$$

From this theorem we can see that the typical configuration in V belonging to N_ρ^- satisfies the following properties if $0<\rho<1/4$: (i) $|\Lambda| \sim 2\rho^{1/2}|V|^{1/2}$; (ii) $|\Theta_-^{\max}| \sim \rho|V|$ asymptotically with probability one as $V \uparrow Z^2$. These properties mean that $\Theta_-^{\max}$ lies in one of the corners of V and the shape of $\Theta_-^{\max}$ is nearly square with the size of $\rho^{1/2}|V|^{1/2}$.

In other cases similar pictures of phase separations are obtained.

4. PROPERTIES OF GIBBS MEASURE IN ISING MODEL WITH PURE BOUNDARY CONDITION

We state some properties of Gibbs measure in Ising model with pure boundary condition which will be used in the sequel, and their proofs are given in Refs. 1 and 2.

Let $\Omega_{V,\pm}$ be the configuration space in V with ($\pm$)-boundary condition. We denote the Gibbs measure on $\Omega_{V,\pm}$ with ($\pm$)-boundary condition by $P_{V,\pm}$. We also denote the expectation value and the variance with respect to $P_{V,\pm}$ by $\langle\cdot\rangle_{V,\pm}$ and $D_{V,\pm}(\cdot)$, respectively.

Lemma 4.1 (Minlos and Sinai). For sufficiently large β we have

$$|\langle N_\pm \rangle_{V,\pm} - \rho^{**}(\beta)|V|| < F_1(\beta)|V|^{1/2} \tag{1}$$

$$D_{V,\pm}(N_\pm) < F_2(\beta)|V| \tag{2}$$

where $\rho^{**}(\beta) = 1-\rho^*(\beta)$ and $\rho^*(\beta), F_1(\beta), F_2(\beta) \sim \exp(-4\beta)$.

The value of $\rho^*(\beta)$ is determined through the correlation function in infinite region. (See Refs. 1 and 2.)

Let W be the subset of V and $P_{W,\pm,c_0}$ be the conditional measure on $\Omega_{W,\pm}$ under the condition that all outer contours are c_0-small. Then the following estimate is given in Ref. 1 (pages 349 and 360):

Lemma 4.2 (Minlos and Sinai). If $|W| > k|V|$, then for sufficiently large β and V we have

$$P_{W,\pm,c_0}\left(\left|N_\pm(\xi;W) - \rho^{**}(\beta)|W|\right| > t|W|^{3/4}\right)$$
$$< c\exp\left[-q(\beta)t^2k^{1/2}|V|^{1/2}\right]$$

where $q(\beta) \sim \exp(-4\beta)$ and c is the absolute constant.

5. ESTIMATE OF $P_V(N_\rho^-)$ FROM BELOW

For a finite set C of Z^2, we put

$$\partial C = \{t \in V\backslash C;\ |t-s| = 1 \text{ for some } s \in C\}$$

$$\partial_{\text{in}} C = \{t \in C;\ |t-s| = 1 \text{ for some } s \in V\backslash C\}$$

$$\operatorname{Int} C = C\backslash\partial_{\text{in}} C \quad \text{and} \quad \bar{C} = C \cup \partial C$$

Let us introduce the correlation function of contours,

$$\rho_V(\overline{\Gamma}_1, \ldots, \overline{\Gamma}_s, \overline{\Delta}_1, \ldots, \overline{\Delta}_k) = P_V(\xi; \{\overline{\Gamma}_1, \ldots, \overline{\Gamma}_s, \overline{\Delta}_1, \ldots, \overline{\Delta}_k\} \subset \xi)$$

By using the Peierls' argument we have the following estimate:

Lemma 5.1. $\rho_V(\overline{\Gamma}_1, \ldots, \overline{\Gamma}_s, \overline{\Delta}_1, \ldots, \overline{\Delta}_k) < \exp[-\beta(\sum_{i=1}^{s}|\Gamma_i| + \sum_{j=1}^{k}|\Delta_j|]$.

The key for the proof of our theorem is the following estimate of $P_V(N_\rho^-)$ from below.

Lemma 5.2. Let us fix $\rho \in (0, 1/4) \cup (1/4, 3/4) \cup (3/4, 1)$ and take the value of β sufficiently large. Then we have the following estimates of $P_V(N_\rho^-)$ for sufficiently large V according to the value of ρ:

(I) $0 < \rho < 1/4$:

$$P_V(N_\rho^-) > c\exp\left\{-\left[2\rho^{1/2}\beta + m(\beta)\right]|V|^{1/2}\right\}$$

(II) $1/4 < \rho < 3/4$:

$$P_V(N_\rho^-) > c\exp\left\{-\left[\beta + m(\beta)\right]|V|^{1/2}\right\}$$

(III) $3/4 < \rho < 1$:

$$P_V(N_\rho^-) > c\exp\left\{-\left[2(1-\rho)^{1/2}\beta + m(\beta)\right]|V|^{1/2}\right\}$$

where $m(\beta) \sim \exp(-4\beta)$ and c is the absolute constant.

Proof of Lemma 5.2. We only treat the case of (I), since the proofs in other cases are very similar. Let V_1 be the square with size of $\rho^{1/2}|V|^{1/2}$ and Γ be the open contour at the corner shown in Fig. 4. Put $\overline{\Gamma} = (\Gamma, +)$ and

$$M(\overline{\Gamma}) = \left\{\xi \in \Omega_V; \Gamma \in \xi^1_{\text{open}}, \overline{\Gamma} : \text{outer}, \xi^2_{\text{open}} = \phi\right\}$$

For each $\xi \in M(\overline{\Gamma})$, put

$$\Lambda_1(\xi) = \text{Int}(V_1 \backslash R'(\xi)) \quad \text{and} \quad \Lambda_2(\xi) = \text{Int}(V_2 \backslash R'(\xi))$$

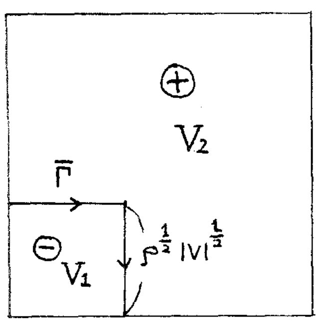

Fig. 4.

where

$$R'(\xi) = \bigcup_{\bar{\Delta} \in \xi^1_{\text{open}} \setminus \{\bar{\Gamma}\}} r(\Delta)$$

and $r(\Delta)$ is the region surrounded by $\bar{\Delta}$. It is clear that $\Lambda_1(\xi)$ has $(-)$-boundary condition and $\Lambda_2(\xi)$ has $(+)$-boundary condition. Put $\lambda(\xi) = (\Lambda_1(\xi), \Lambda_2(\xi))$. When the following condition is satisfied, configuration $\xi \in M(\bar{\Gamma})$ is called γ-regular $(1/2 < \gamma < 1)$.

$$||\Lambda_1(\xi)| - \rho|V|| < g(\beta)|V|^{\gamma}$$
$$||\Lambda_2(\xi)| - (1-\rho)|V|| < g(\beta)|V|^{\gamma}$$

Put

$$\Pi_\gamma = \{(W_1, W_2);\ \lambda(\xi) = (W_1, W_2) \text{ for some } \gamma\text{-regular } \xi \in M(\bar{\Gamma})\}$$

For each $\lambda \in \Pi_\gamma$, put

$$M_{\bar{\Gamma}}(\lambda) = \{\xi \in M(\bar{\Gamma});\ \lambda(\xi) = \lambda\}$$

We denote the totality of γ-regular configurations by $M_\gamma(\bar{\Gamma})$.

By a standard argument, we have

$$P_V(N_\rho^-) > \sum_{\lambda \in \Pi_\gamma} P_V(M_{\bar{\Gamma}}(\lambda)) P_\lambda(N_\rho^-) \tag{5.1}$$

where $P_\lambda(\cdot) = P_V(\cdot \mid M_{\bar{\Gamma}}(\lambda))$.

We estimate $P_\lambda(N_\rho^-)$ and $P_V(M_\gamma(\bar{\Gamma}))$ separately from below. For each $\lambda = (W_1, W_2) \in \Pi_\gamma$, put

$$A_\lambda = \{\xi \in M_{\bar{\Gamma}}(\lambda);\ |N_+(\xi; W_2) - \rho^{**}(\beta)|W_2|| < F_2(\beta)^{1/3}|W_2|^{1/2}$$
$$\text{and } |N_-(\xi; W_1) - \rho^{**}(\beta)|W_1|| < F_2(\beta)^{1/3}|W_2|^{1/2}\}$$

where $F_2(\beta)$ is the function given in Lemma 4.1. For sufficiently large β and V, we have $A_\lambda \subset M_{\bar{\Gamma}}(\lambda) \cap N_\rho^-$.

By Lemma 4.1 and Chevyshev's inequality, we have

$$\begin{aligned} P_\lambda(N_\rho^-) &> P_\lambda(A_\lambda) \\ &= P_{W_1,-}\left(\left|N_-(\xi; W_1) - \rho^{**}(\beta)|W_1|\right| < F_2(\beta)^{1/3}|W_1|^{1/2}\right) \\ &\quad \times P_{W_2,+}\left(\left|N_+(\xi; W_2) - \rho^{**}(\beta)|W_2|\right| < F_2(\beta)^{1/3}|W_2|^{1/2}\right) \\ &> 1 - u(\beta) \end{aligned} \tag{5.2}$$

where $u(\beta) \downarrow 0$ exponentially fast as $\beta \to \infty$.

Next we estimate $P_V(M_\gamma(\bar{\Gamma}))$. For each $\lambda = (W_1, W_2) \in \Pi_\gamma$, put

$$N(\lambda) = \{\xi \in \Omega_V;\ \xi^2_{\text{open}} = \phi \text{ and } V \setminus R(\xi) = \bar{W}_1 \cup \bar{W}_2\}$$

where

$$R(\xi) = \bigcup_{\bar{\Delta} \in \xi^1_{\text{open}}} r(\Delta)$$

By setting $Z(M_{\bar{\Gamma}}(\lambda)) = \sum_{\xi \in M_{\bar{\Gamma}}(\lambda)} \exp[-\beta E_V(\xi)]$, we have

$$P_V\big(M_\gamma(\bar{\Gamma})\big) = \frac{\sum_{\lambda \in \Pi_\gamma} Z(M_{\bar{\Gamma}}(\lambda))}{Z_V}$$

$$= P_V(N) \frac{\sum_{\lambda \in \Pi_\gamma} Z(M_{\bar{\Gamma}}(\lambda))}{\sum_{\lambda \in \Pi_\gamma} Z(N(\lambda))} \tag{5.3}$$

where $E_V(\xi) = \sum_{\bar{\Gamma} \in \xi} |\Gamma|$ and $N = \bigcup_{\lambda \in \Pi_\gamma} N(\lambda)$.

For each $\lambda = (W_1, W_2) \in \Pi_\gamma$, put $R_\lambda = V \backslash (\overline{W}_1 \cup \overline{W}_2)$. From the symmetry in $(+)$ and $(-)$, we have

$$Z(N(\lambda)) = 2 Z_{W_1 W_2, +} Z_{R_\lambda, +} \exp(-\beta |\partial R_\lambda|) \tag{5.4}$$

As for $Z(M_{\bar{\Gamma}}(\lambda))$ we have the following expression:

$$Z(M_{\bar{\Gamma}}(\lambda)) = Z_{W_1,+} Z_{W_2,+} Z_{R_\lambda,+} \exp(-\beta|\Gamma| - \beta|\partial R_\lambda|) \tag{5.5}$$

To estimate $Z(M_{\bar{\Gamma}}(\lambda))/Z(N(\lambda))$ we use the following lemma which is proved in Ref. 2 by using the estimates of correlation functions:

Lemma 5.3 (Minlos and Sinai). For sufficiently large β, W_1, and W_2, the following estimate holds:

$$\frac{Z_{W_1,+} Z_{W_2,+}}{Z_{W_1 \cup W_2,+}} > \big[1 - v(\beta)\big] \exp\big[-m(\beta) |W_1 \cup W_2|^{1/2}\big]$$

where $v(\beta) \downarrow 0$ and $m(\beta) \sim \exp(-4\beta)$ as $\beta \to \infty$.

From this lemma we have

$$\frac{Z(M_{\bar{\Gamma}}(\lambda))}{Z(N(\lambda))} = \frac{\exp(-\beta|\Gamma|) Z_{W_1,+} Z_{W_2,+}}{2 Z_{W_1 \cup W_2,+}}$$

$$> \tfrac{1}{2}\big[1 - v(\beta)\big] \exp\big\{-\big[2\beta\rho^{1/2} + m(\beta)\big]|V|^{1/2}\big\}$$

From this estimate and (5.3), we have

$$P_V\big(M_\gamma(\bar{\Gamma})\big) > \tfrac{1}{2}\big[1 - v(\beta)\big] P_V(N) \exp\big\{-\big[2\beta\rho^{1/2} + m(\beta)\big]|V|^{1/2}\big\} \tag{5.6}$$

We have only to obtain the estimate of $P_V(N)$ from below for the proof of Lemma 5.2. Remark that

$$N^c = N^{(1)} \cup N^{(2)} \cup N^{(3)}$$

$$N^{(1)} = \big\{\xi;\ \xi^2_{\text{open}} \neq \phi\big\}$$

$$N^{(2)} = \big\{\xi;\ \xi^2_{\text{open}} = \phi |R(\xi)| > g(\beta)|V|^\gamma\big\}$$

$$N^{(3)} = \big\{\xi;\ \xi^2_{\text{open}} = \phi\, \exists \bar{\Gamma}_1 \in \xi^1_{\text{open}} \text{ s.t. } \Gamma_1 \text{ touches } \Gamma\big\}$$

From Lemma 5.1 through an elementary but tedious calculation we have

$$P_V(N^c) < \text{const} \cdot \exp(-3\beta) \tag{5.7}$$

By combining (5.2), (5.6), and (5.7) we complete the proof of Lemma 5.2.

6. PROOF OF THEOREM

In this section we give the proof of the theorem. First we estimate the length of phase boundary $|\Lambda|$ with respect to $P_V(\cdot)$. Let $M_{T,k}$ be the set of configurations given by

$$M_{T,k} = \{\xi \in \Omega_V;\ |\Lambda(\xi)| = T \text{ and } \#(\Lambda(\xi)) = k\}$$

where $\#(\Lambda(\xi))$ is the number of lines in $\Lambda(\xi)$. By using the similar way of counting the lines to the proof of Lemma 5.1 in Ref. 1, we have

$$P_V(m_{T,k}) < |V|^{k_2 k_3 T} \exp(-\beta T)$$

Put $M_T = \{\xi;\ |\Lambda(\xi)| = T\}$. For each $\xi \in M_T$, the length of each line in $\Lambda(\xi)$ is larger than $c_0 \ln|V|$. So the number of lines in $\Lambda(\xi)$ must be smaller than $T/c_0 \ln|V|$. Hence we have the following estimate of $P_V(M_T)$:

$$\begin{aligned} P_V(M_T) &< \sum_{1 \leqslant k \leqslant T/c_0 \ln|V|} (2|V|)^k \exp[-(\beta - \ln 3)T] \\ &< \exp[-(\beta - \ln 3)T] \int_0^{T/c_0 \ln|V|} \exp(k \ln 2|V|)\, dk \\ &< (2/\ln|V|) \exp[-(\beta - 2/c_0)T] \end{aligned}$$

Hence, we have the following estimate:

Lemma 6.1. $P_V(|\Lambda(\xi)| > T) < (4/\ln|V|)\exp[-(\beta - 2/c_0)T]$.

From this lemma and Lemma 5.2, we have the following estimate of $|\Lambda(\xi)|$ with respect to $P_{V,\rho}(\cdot)$.

Lemma 6.2. Let us fix $\rho \in (0, 1/4) \cup (1/4, 3/4) \cup (3/4, 1)$ and take the value of β sufficiently large, then we have the following estimates for sufficiently large V:

(1) $0 < \rho < 1/4$

$$\begin{aligned} &P_{V,\rho}(|\Lambda| > (2\rho^{1/2} + k/\beta)|V|^{1/2}) \\ &\quad < (4/\ln|V|)\exp\{-[k - (4/c_0)\rho^{1/2} - (2k/c_0\beta) - m(\beta)]|V|^{1/2}\} \end{aligned}$$

(2) $1/4 < \rho < 3/4$

$$\begin{aligned} &P_{V,\rho}(|\Lambda| > (\rho^{1/2} + k/\beta)|V|^{1/2}) \\ &\quad < (4/\ln|V|)\exp\{-[k - (4/c_0)\rho^{1/2} - (2k/c_0\beta) - m(\beta)]|V|^{1/2}\} \end{aligned}$$

(3) $3/4 < \rho < 1$

$$P_{V,\rho}\left(|\Lambda| > \left[2(1-\rho)^{1/2} + k/\beta\right]|V|^{1/2}\right)$$
$$< (4/\ln|V|)\exp\left\{-\left[k - (4/c_0)\rho^{1/2} - (2k/c_0\beta) - m(\beta)\right]|V|^{1/2}\right\}$$

If we put $k_0 = 4/c_0 + 1$, then $k_0 - (4/c_0)\rho^{1/2} - (2k/c_0\beta) - m(\beta) > 0$ for sufficiently large β. Hence we have the following estimate in the case of $\rho \in (0, 1/4)$:

$$\lim_{V\uparrow Z^2} P_{V,\rho}\left(|\Lambda| > (2\rho^{1/2} + k_0/\beta)|V|^{1/2}\right) = 0$$

For simplicity we denote this estimate by

$$|\Lambda| < (2\rho^{1/2} + k_0/\beta)|V|^{1/2} \qquad \text{asymptotically with } P_{V,\rho}\text{-prob. } 1 \tag{6.1}$$

Similarly we have

$$|\Lambda| < (\rho^{1/2} + k_0/\beta)|V|^{1/2} \qquad \text{asymptotically with } P_{V,\rho}\text{-prob. } 1 \tag{6.2}$$

if $\rho \in (1/4, 3/4)$ and

$$|\Lambda| < \left[2(1-\rho)^{1/2} + k_0/\beta\right]|V|^{1/2}$$
$$\text{asymptotically with } P_{V,\rho}\text{-prob. } 1 \tag{6.3}$$

if $\rho \in (3/4, 1)$.

From now on we give the proof of the first assertion of the theorem. The main tool for the proof is the estimate of Lemma 4.2.

First recall the fact that V is divided into three parts, $(+)$-phase Θ_+, $(-)$-phase Θ_-, and R_{c_0} by phase boundaries and the elements of ξ^1_{open}. From the definition of c_0-smallness we have

$$|R_{c_0}(\xi)| < 2c_0|V|^{1/2}\ln|V| \tag{6.4}$$

We also remark that the boundaries of Θ_+ and Θ_- are occupied by $(+)$-spins and $(-)$-spins, respectively, and that all outer contours in Θ_+ and Θ_- are c_0-small closed contours. So we can apply the estimate of Lemma 4.2 to $N_\pm(\xi; \Theta_\pm)$.

Let $r(\beta)$ be the function of β satisfying $r(\beta) \sim \exp(-3\beta)$. Consider the following set of configurations:

$$C_1 = \left\{\xi \in N_\rho^-;\ |\Theta_-(\xi)| > \left[\rho + r(\beta)\right]|V|\right\}$$

We prove in the following

$$\lim_{V\uparrow Z^2} P_{V,\rho}(C_1) = 0 \qquad \text{for sufficiently large } \beta \tag{6.5}$$

From Lemma 4.2, we have

$$\lim_{V\uparrow Z^2} P_{V,\rho}(C_1 \cap D_1) = 0$$

Furthermore if $\xi \in C_1 \cap D_1^c$, then $N_-(\xi;\Theta_-) > \rho|V| + 1/2r(\beta)|V|$ for sufficiently large β and V. Consequently $C_1 \cap D_1^c = \phi$. Hence, we have obtained the proof of (6.5).

Put

$$C_2 = \{\xi \in N_\rho^-;\ |\Theta_-| < [\rho - r(\beta)]|V|\}$$

Next we prove

$$\lim_{V\uparrow Z^2} P_{V,\rho}(C_2) = 0 \tag{6.6}$$

For each $\xi \in C_2$, $|\Theta_+| > [1-\rho+1/2r(\beta)]|V|$ for sufficiently large β and V. From this fact we can prove (6.6) by using the same argument as above. Hence we obtain the following lemma.

Lemma 6.3. Let us fix the value of β sufficiently large. Then the following estimates are satisfied:

(i) $$\lim_{V\uparrow Z^2} P_{V,\rho}(\big||\Theta_-| - \rho|V|\big| > r(\beta)|V|) = 0$$

(ii) $$\lim_{V\uparrow Z^2} P_{V,\rho}(\big||\Theta_+| - (1-\rho)|V|\big| > r(\beta)|V|) = 0$$

where $r(\beta)$ is the function of β satisfying $r(\beta) \sim \exp(-3\beta)$.

Finally we prove

$$\lim_{V\uparrow Z^2} P_{V,\rho}(|\Theta_-^{\max}| < [\rho - k(\beta)]|V|) = 0 \tag{6.7}$$

where $k(\beta) = (5k_0/3\beta)^2$.

Consider the following set of configurations:

$$C_3 = \{\xi \in N_\rho^-;\ |\Theta_-^{\max}| < [\rho - k(\beta)]|V|\}$$

Put $\Theta_-^{\text{rem}} = \Theta_- \setminus \Theta_-^{\max}$. If V is sufficiently large,

$$|\Theta_-^{\text{rem}}(\xi)| > \tfrac{16}{25}k(\beta)|V| \qquad \text{for each} \quad \xi \in C_3 \tag{6.8}$$

Then we have the following estimate as to the minimal length of $\Lambda(\xi)$:

$$\begin{aligned}|\Lambda(\xi)| &> 2[\rho - k(\beta)]^{1/2}|V|^{1/2} + 2(4/5)k(\beta)^{1/2}|V|^{1/2} \\ &> (2\rho^{1/2} + \tfrac{7}{5}k(\beta)^{1/2})|V|^{1/2}\end{aligned} \tag{6.9}$$

From (6.1) and (6.9), we have

$$|\Theta_-^{\max}| > [\rho - k(\beta)]|V| \qquad \text{asymptotically with } P_{V,\rho}\text{-prob. } 1$$

Hence the first assertion of the theorem is proved. The second assertion is evident from Lemma 4.2.

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